Exercice 1: $U_n = \sum_{k=1}^{n} \frac{1}{\sqrt{(n+k-1)(n+k)}}$ pour $n \geq 1$

(a) montrons que pour tout $n \geq 1$ et $k \geq 1$ on a:

$$\frac{1}{n+k} \leq \frac{1}{\sqrt{(n+k-1)(n+k)}} \leq \frac{1}{n+k-1}$$

on a $(n+k-1)^2 \leq (n+k)^2$

$$0 < (n+k-1)(n+k-1) \leq (n+k-1)(n+k) \leq (n+k)^2$$

$$(n+k-1)^2 \leq (n+k-1)(n+k) \leq (n+k)^2$$

$$0 \leq \frac{1}{(n+k)^2} \leq \frac{1}{(n+k-1)(n+k)} \leq \frac{1}{(n+k-1)^2}$$

$x \longrightarrow \sqrt{x}$ est croissante

$$\sqrt{\frac{1}{(n+k)^2}} \leq \sqrt{\frac{1}{(x+k-1)(n+k)}} \leq \sqrt{\frac{1}{(n+k-1)^2}}$$

$$\frac{\sqrt{1}}{\sqrt{(n+k)^2}} \leq \frac{\sqrt{1}}{\sqrt{(x+k-1)(n+k)}} \leq \frac{\sqrt{1}}{\sqrt{(n+k-1)^2}} \quad n \geq 1;\ k \geq 1$$

$$\frac{1}{|n+k|} \leq \frac{1}{\sqrt{(n+k-1)(n+k)}} \leq \frac{1}{|n+k-1|}$$

$$\boxed{\frac{1}{n+k} \leq \frac{1}{\sqrt{(n+k-1)(n+k)}} \leq \frac{1}{n+k-1}}$$

(b) on a $\sum_{k=1}^{n} \frac{1}{n+k} \leq \sum_{k=1}^{n} \frac{1}{\sqrt{(n+k-1)(n+k)}} \leq \sum_{k=1}^{n} \frac{1}{n+k-1}$

$$\sum_{k=1}^{n} \frac{1}{n+k} = \sum_{k=1}^{n} (x_{k+1} - x_k)\, f(x_k)$$

$$\sum_{k=1}^{n} \frac{1}{n+k} = \sum_{k=1}^{n} \frac{1}{n} \frac{1}{1+k/n}$$

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$x_{k+1} - x_k = \frac{1}{n}$; $a=0$; $b=1$

d'où $f(x) = \frac{1}{1+x}$, f continue sur $[0;1]$ donc :

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{n+k}=\lim_{n\to+\infty}\frac{1}{n}\frac{1}{1+k/n}=\int_0^1\frac{1}{1+x}dx$$

$$=\left[\ln(1+x)\right]_0^1=\ln 2$$

donc $\boxed{\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{n+k}=\ln 2}$

on a $\sum_{k=1}^{n}\frac{1}{n+k-1}=\sum_{k=1}^{n}\frac{1}{n}\frac{1}{1+\frac{(k-1)}{n}}=\sum_{k=0}^{n-1}\frac{1}{1+\frac{k}{n}}$

$x_{k+1}-x_k=\frac{1}{n}$; $f(x)=\frac{1}{1+x}$

$a=0$; $b=1$. f continu sur $[0;1]$ donc

$$\lim_{n\to+\infty}\sum_{k=0}^{n-1}\frac{1}{n}\frac{1}{1+k/n}=\int_0^1\frac{1}{1+x}dx=\boxed{\ln 2}$$

Conclusion : on a $\sum_{k=1}^{n}\frac{1}{n+k}\leq u_n\leq\sum_{k=1}^{n}\frac{1}{n+k-1}$

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{n+k}\leq\lim_{n\to+\infty}u_n\leq\lim_{n\to+\infty}\sum_{k=1}^{n}\frac{1}{n+k-1}$$

$$\Rightarrow\quad \ln 2\leq\lim_{n\to+\infty}u_n\leq\ln 2$$

d'où $\boxed{\lim_{n\to+\infty}u_n=\ln 2}$

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Exercice 2 : $V_n=\frac{1}{n^3}\sum_{k=1}^{n}k\sqrt{k^2+n^2}$

on a $V_n=\frac{1}{n^3}\sum_{k=1}^{n}kn\sqrt{\left(\frac{k}{n}\right)^2+1}=\sum_{k=1}^{n}\frac{k}{n^2}\sqrt{\left(\frac{k}{n}\right)^2+1}$

$$V_n=\sum_{k=1}^{n}\frac{1}{n}\frac{k}{n}\sqrt{\left(\frac{k}{n}\right)^2+1}$$

on a $x_k=\frac{k}{n}$; $x_{k+1}-x_k=\frac{1}{n}$; $a=0$; $b=1$

$f(x)=x\sqrt{x^2+1}$. $f(x)$ est continue sur $[0;1]$

donc $\lim_{x\to+\infty}f(x)=\frac{1}{2}\int_0^1 2x\sqrt{x^2+1}\,dx$

$$\lim_{n\to+\infty}f(x)=\frac{1}{2}\left[\frac{(x^2+1)^{3/2}}{3/2}\right]_0^1=\frac{1}{2}\frac{2}{3}\left[2^{3/2}-1\right]=\frac{1}{3}(2\sqrt{2}-1)$$

d'où $\lim_{n\to+\infty}V_n=\int_0^1 x\sqrt{x^2+1}\,dx=\boxed{\frac{1}{3}\left[2\sqrt{2}-1\right]}$

$$\ast\ a_n = \frac{1}{n}\left(\frac{(2n)!}{n!}\right)^{1/n} \qquad \left[x^y = e^{\ln(x^y)} = e^{y\ln x}\right]$$

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on a $\ln(a_n) = \ln\left(\frac{1}{n}\left(\frac{(2n)!}{n!}\right)^{1/n}\right)$

$$= \ln(1/n) + \ln\left(\left(\frac{(2n)!}{n!}\right)^{1/n}\right)$$

$$= \ln(1/n) + \frac{1}{n}\ln\left(\frac{(2n)!}{n!}\right)$$

on a $\frac{(2n)!}{n!} = \frac{1\times 2\times 3\times \cdots n\times(n+1)\times\cdots(2n)}{1\times 2\times 3\times\cdots\times n}$

$$= (n+1)(n+2)(n+3)\cdots(n+n)$$

$$\ln(a_n) = \ln\left(\frac{1}{n}\right) + \frac{1}{n}\ln\left((n+1)(n+2)\cdots(n+n)\right)$$

$$= \ln(1/n) + \frac{1}{n}\sum_{k=1}^{n}\ln(n+k)$$

$$\ln(a_n) = \ln(1/n) + \frac{1}{n}\sum_{k=1}^{n}\ln(n(1+k/n))$$

$$= \ln(1/n) + \frac{1}{n}\sum_{k=1}^{n}\ln(n) + \ln(1+k/n)$$

$$\ln(a_n) = \ln(1/n) + \frac{1}{n}\sum_{k=1}^{n}\ln(n) + \frac{1}{n}\sum_{k=1}^{n}\ln(1+k/n)$$

on a $\sum_{k=1}^{n}\ln(n) = \ln(n) + \ln(n) - - - + \ln(n) = n\ln(n)$

donc $\ln(a_n) = \ln\left(\frac{1}{n}\right) + \ln(n) + \frac{1}{n}\sum_{k=1}^{n}\ln(1+k/n)$

$$\ln(a_n) = \ln(1) + \sum_{k=1}^{n}\frac{1}{n}\ln(1+k/n)$$

on a $a=0$; $b=1$; $f(x) = \ln(1+x)$; f est continue sur $[0,1]$

$$\lim_{n\to+\infty}\ln(a_n) = \int_0^1 \ln(1+x)\,dx = \int_1^2 \ln z\,dz = [z\ln z - z]_1^2$$

$$= 2\ln 2 - 2 + 1 = 2\ln 2 - 1$$

on a f continue en $x_0 \Leftrightarrow \lim_{x\to x_0} f(x) = f(x) = f(\lim_{x\to x_0} x)$

$\ln$ est continue donc

$$\lim_{n\to+\infty} \ln(a_n) = \ln\left(\lim_{n\to+\infty} a_n\right) = 2\ln(2)-1$$

$$\lim_{n\to+\infty} a_n = e^{2\ln 2 - 1} \Leftrightarrow \boxed{\lim_{n\to+\infty} a_n = \frac{4}{e}}$$

Exercice 3 : (a) $I = \int \cos(t)\, e^t \, dt$

on a $u = \cos(t) \Rightarrow u' = -\sin(t)$

$v' = e^t \Rightarrow v = e^t$

$$I = \cos(t)\, e^t - \int -\sin(t)\, e^t \, dt = \cos(t)\, e^t + \int \sin(t)\, e^t \, dt$$

$u = \sin t \Rightarrow u' = \cos(t)$

$v' = e^t \Rightarrow v = e^t$

$$I = \cos(t)\, e^t + \sin(t)\, e^t - \int \cos(t)\, e^t \, dt$$

donc $2I = (\cos(t) - \sin(t))e^t$

$$\boxed{I = \frac{1}{2}(\cos(t) + \sin(t))e^t}$$

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(b) déduisant $J = \int \frac{e^{\arctan(t)}}{(1+t^2)^{3/2}} dt$

on a $x = \arctan(t) \Leftrightarrow t = \tan(x)$

$dt = (1 + \tan^2(x))dx$

donc $J = \int \frac{e^x}{(1+\tan^2(x))^{3/2}} (1+\tan^2(x))\, dx$

donc $J = \int \frac{e^x}{\sqrt{1+\tan^2(x)}} dx = \int \cos(x)\, e^x \, dx$

donc $\boxed{J = I}$

(3) Analyse (2) serie (1)

Exercice 4:

a) $f(t) = \frac{1}{t(1+t^2)} = \frac{a}{t} + \frac{bt+c}{1+t^2}$ Methode (1)

$f(t)\times t = \frac{1}{1+t^2}/_{t=0} = a + \frac{(bt+c)t}{1+t^2}/_{t=0} = \boxed{a=1}$

$\lim_{t\to+\infty} f(t)\times t = \lim_{t\to+\infty} a + \frac{bt^2+ct}{1+t^2} = a+b = \lim_{t\to+\infty} \frac{1}{1+t^2} = 0$

donc $a+b=0 \iff \boxed{b=-1}$

$\boxed{t=1}$ $\frac{1}{1(1+1^2)} = \frac{a}{1} + \frac{b+c}{1+1}$

$\iff \frac{1}{2} = 1 + \frac{-1+c}{2} \Rightarrow 1 = 2-1+c \iff \boxed{c=0}$

donc $\boxed{f(t) = \frac{1}{t} - \frac{t}{1+t^2}}$

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Methode (2)

$f(t) = \frac{1}{t(1+t^2)} = \frac{(1+t^2)a + t(bt+c)}{t(1+t^2)} = \frac{(a+b)t^2+ct+a}{t(1+t^2)}$

$\boxed{a=1} \Rightarrow \boxed{c=0}$; $a+b=0 \Rightarrow \boxed{b=-1}$

b) $J(t) = \int \frac{t\ln(t)}{(1+t^2)^2}$

on pose $u = \ln(t)$ et $v' = \frac{t}{(1+t^2)^2} = \frac{1}{2}(2t(1+t^2)^{-2})$

$u' = \frac{1}{t}$ et $v = \frac{1}{2}\frac{(1+t^2)^{-2+1}}{-2+1}$

donc $J(t) = -\frac{1}{2}\frac{\ln(t)}{1+t^2} + \frac{1}{2}\int \frac{1}{t(1+t^2)}dt$

$= -\frac{1}{2}\frac{\ln(t)}{1+t^2} + \frac{1}{2}\int\frac{dt}{t} - \frac{1}{2}\int\frac{t}{1+t^2}dt$

$\Rightarrow \boxed{J(t) = -\frac{1}{2}\frac{\ln(t)}{1+t^2} + \frac{1}{2}\ln(t) - \frac{1}{4}\ln(1+t^2) + cte}$

Exercice 5

a) $f(x) = \frac{x^2}{(1+x)^2(2+x)} = \frac{a}{1+x} + \frac{b}{(1+x)^2} + \frac{c}{2+x}$

$f(x) = \frac{a(2+x)(1+x) + b(2+x) + c(1+x)^2}{(1+x)^2}$

$f(x) = \frac{x^2(a+c) + x(3a+b+2c) + 2a+2b+c}{(1+x)^2(2+x)}$

$a+c = 1$ $\quad\quad 3(a+c=1)$

$(3a+b+2c) = 0$ $\quad\quad -4a-3c=0$

$2a+2b+c=0$ $\quad\quad \Rightarrow -a = 3 \Rightarrow \boxed{a=-3}$

$\Rightarrow c = 1-a = 1-(-3) = 4 \Leftrightarrow \boxed{c=4}$

$3\times(-3)+b+2\times 4 = 0 \Rightarrow \boxed{b=1}$

donc $\boxed{f(x) = \frac{-3}{1+x} + \frac{1}{(1+x)^2} + \frac{4}{2+x}}$

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b) $g(t) = \int \frac{t^4}{(1+t^2)(2+t^2)} dt$

on pose $x = t^2 \Rightarrow \frac{t^4}{(1+t^2)(2+t^2)} = \frac{x^2}{(1+x)^2(2+x)}$

$\Rightarrow \frac{-3}{1+x} + \frac{1}{(1+x)^2} + \frac{4}{(2+x)}$

$\Rightarrow \frac{-3}{1+t^2} + \frac{1}{(1+t^2)^2} + \frac{4}{(2+t^2)}$

$g(t) = -3\int \frac{dt}{1+t^2} + \int \frac{dt}{(1+t^2)^2} + \frac{4}{2+t^2}$

$g(t) = -3\arctan(t) + \frac{1}{2}\int \frac{2t\,dt}{(1+t^2)^2 t} + \frac{4}{2}\sqrt{2}\int \frac{d\left(\frac{t}{\sqrt{2}}\right)}{1+\left(t/\sqrt{2}\right)^2}$

$g(t) = -3\arctan(t) + \frac{1}{2}\int \frac{2t}{t(1+t^2)^2} + 2\sqrt{2}\arctan\left(t/\sqrt{2}\right)$

on calcule $\int \frac{2t}{t(1+t^2)^2} dt$

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4 **Analyse 2** **serie 1**:

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on pose $u = \frac{1}{t}$ et $v' = \frac{2t}{(1+t^2)^2}$

$u' = -\frac{1}{t^2}$ et $v = -\frac{1}{1+t^2}$

donc $\int \frac{2t}{t(1+t^2)^2} = \frac{-1}{t(1+t^2)} - \int \frac{dt}{t^2} + \int \frac{dt}{1+t^2}$

car $\frac{1}{t^2(1+t^2)} = \frac{1}{t^2} - \frac{1}{1+t^2}$

$\Rightarrow \boxed{\int \frac{2t}{t(1+t^2)^2} dt = \frac{-1}{t(1+t^2)} + \frac{1}{t} + \arctan(t) + cte}$

$= \frac{-1+(1+t^2)}{t(1+t^2)} + \arctan(t) + cte$

$\boxed{\int \frac{2t}{t(1+t^2)^2} dt = \frac{t}{1+t^2} + \arctan(t) + cte}$

donc $g(t) = -3\arctan(t) + \frac{1}{2}\frac{t}{1+t^2} + \frac{1}{2}\arctan(t) + 2\sqrt{2}\,\mathrm{arctg}\left(\frac{t}{\sqrt{2}}\right) + c$

$\boxed{g(t) = -\frac{5}{2}\arctan(t) + 2\sqrt{2}\arctan\left(\frac{t}{\sqrt{2}}\right) + \frac{1}{2}\frac{t}{1+t^2} + cte}$

c) calculons alors $\int_0^{\pi/4} \frac{\sin^4(x)}{1+\cos^2(x)} dx$

$\frac{\sin^4(x)/\cos^4(x)}{\frac{1+\cos^4(x)}{\cos^4(x)}} = \frac{t^4}{\frac{\cos^2(x)}{\cos^4(x)}\left(\frac{1}{\cos^2(x)}+1\right)}$

on pose $t = \tan(x)$

$\frac{1}{\cos^2(x)} = 1+\tan^2(x)$

$dt = (1+\tan^2(x))\,dx$

$\frac{dt}{1+t^2} = dx$

donc $A = \int_0^1 \frac{t^4}{(1+t^2)(2+t^2)} \times \frac{dt}{(1+t^2)}$

$A = \int_0^1 \frac{t^4}{(1+t^2)^2(2+t^2)} dt = I(1) - I(0)$

$\boxed{A = -\frac{5}{2}\times\frac{\pi}{4} + 2\sqrt{2}\arctan\left(\frac{\sqrt{2}}{2}\right) + \frac{1}{2}\times\frac{1}{2} - 0 + 0 + 0}$

Exercice 6 • $I_1 = \int_1^2 \ln(x+\sqrt{1+x^2})\,dx$

$$I_1 = \int_1^2 1\times \ln(x+\sqrt{1+x^2})\,dx = \left[x\ln(x+\sqrt{1+x^2})\right]_1^2 - \int_1^2 x\times\frac{1+\frac{2x}{2\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}\,dx$$

$$I_1 = \left[x\ln(x+\sqrt{1+x^2})\right]_1^2 - \int_1^2 x + \frac{\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}\,dx$$

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$$I_1 = \left[x\ln(x+\sqrt{1+x^2})\right]_1^2 - \int_1^2 \left(x + \frac{1}{\sqrt{1+x^2}}\right)dx$$

$$I_1 = \left[x\ln(x+\sqrt{1+x^2})\right]_1^2 - \left[\frac{x^2}{2}\right]_1^2 - \left[\ln(x+\sqrt{x^2+1})\right]_1^2$$

• $I_2 = \int_0^1 (\arccos(x))^2\,dx$

on pose $t = \arccos(x)$; $\cos(t) = x$; $dx = -\sin(t)\,dt$

$$I_2 = \int_{\pi/2}^{0} t^2 \times -\sin(t)\,dt = \int_0^{\pi/2} t^2 \sin(t)\,dt$$

$u = t^2$; $u' = 2t$; $v' = \sin(t)$; $v = -\cos(t)$

$$I_2 = \left[-t^2\cos(t)\right]_0^{\pi/2} + \int_0^{\pi/2} 2t\cos(t)\,dt = \int_0^{\pi/2} 2t\cos t\,dt$$

$u = 2t$; $u' = 2$; $v' = \cos(t)$; $v = \sin(t)$

$$I_2 = \left[2t\sin(t)\right]_0^{\pi/2} - \int_0^{\pi/2} 2\sin(t)\,dt$$

donc $I_2 = \pi + \left[2\cos(t)\right]_0^{\pi/2} \iff \boxed{I_2 = \pi - 2}$

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Analyse 2 — serie 1

- $I_3 = \int \dfrac{\sin(x)}{\cos(x)(1+\cos^2(x))}\,dx$

on a $t = \tan(x)$

on a $\dfrac{\sin(x)}{\cos(x)(1+\cos^2(x))} = \dfrac{t}{\cos^2(x)\left(\frac{1}{\cos^2(x)}+1\right)} = \dfrac{(1+t^2)t}{1+t^2+1}$

$= \dfrac{t(1+t^2)}{2+t^2}$

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$t = \tan(x)$; $dt = (1+\tan^2(x))\,dx$

$dx = \dfrac{dt}{1+t^2}$ d'où $I_3 = \int \dfrac{t(1+t^2)}{2+t^2}\,\dfrac{dt}{1+t^2} = \dfrac{1}{2}\int \dfrac{2t}{2+t^2}$

$I_3 = \dfrac{1}{2}\ln(2+t^2) + cte = \dfrac{1}{2}\ln(2+\tan^2(x)) + cte$

$\Rightarrow \boxed{I_3 = \dfrac{1}{2}\ln\left(2+\tan^2(x)\right) + cte}$

Exercice 7 :

$J_n = \int_0^1 x^n \sin(\pi x)\,dx$

a) on a $J_{n+2} = \int_0^1 x^{n+2}\sin(\pi x)\,dx$

$u = x^{n+2}$; $u' = (n+2)x^{n+1}$; $v' = \sin(\pi x)$; $v = -\dfrac{\cos(\pi x)}{\pi}$

$J_{n+2} = \left[-x^{n+2}\dfrac{\cos(\pi x)}{\pi}\right]_0^1 + \int_0^1 \dfrac{(n+2)}{\pi}x^{n+1}\cos(\pi x)\,dx$

$J_{n+2} = \dfrac{1}{\pi} + \dfrac{n+2}{\pi}\int_0^1 x^{n+1}\cos(\pi x)\,dx$

on a $u = x^{n+1}$; $u' = (n+1)x^n$; $v' = \cos(\pi x)$; $v = \dfrac{\sin(\pi x)}{\pi}$

$J_{n+2} = \dfrac{1}{\pi} + \dfrac{n+2}{\pi}\left[\left(\dfrac{x^{n+1}\sin(\pi x)}{\pi}\right]_0^1 - \int_0^1 \dfrac{n+1}{\pi}x^n\sin(\pi x)\,dx\right)$

$J_{n+2} = \dfrac{1}{\pi} + \dfrac{n+2}{\pi}\left(-\dfrac{n+1}{\pi}J_n\right)$

d'où $\boxed{J_{n+2} = \dfrac{1}{\pi} - \dfrac{(n+2)(n+1)}{\pi^2}J_n}$

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b) calculons la limite de $n^2 J_n$:

$$\lim_{n\to+\infty} n^2 J_n = \lim_{n\to+\infty} n^2 \int_0^1 x^n \sin(\pi x)\,dx$$

on a $\frac{(n+2)(n+1)}{\pi^2} J_n = \frac{1}{\pi} - J_{n+2}$

$$\Rightarrow n^2 J_n = \frac{n^2\pi}{(n+2)(n+1)} - \frac{n^2\pi^2}{(n+2)(n+1)} J_{n+2}$$

on a $\lim_{n\to+\infty} n^2 J_n = \ell \Rightarrow \lim_{n\to+\infty} (n+2)^2 J_{n+2} = \ell$

$$\Rightarrow \underbrace{n^2 J_n}_{\xrightarrow[n\to+\infty]{} \ell} = \underbrace{\frac{n^2\pi}{(n+2)(n+1)}}_{\xrightarrow[n\to+\infty]{} \pi} - \underbrace{\frac{n^2\pi^2}{(n+2)(n+1)(n+2)^2}(n+2)^2 J_{n+2}}_{\xrightarrow[n\to+\infty]{} 0\times\ell}$$

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donc $\boxed{\lim_{n\to+\infty} n^2 J_n = \pi}$

Exercice 8 : $G(x) = \int_0^{x^2} \sqrt{1+t^2}\,dt \;;\; x\in\mathbb{R}$

a)
on a $f(t) = \sqrt{1+t^2}$ continue sur $\mathbb{R}$ admet une primitive sur $[0;x^2]$ notée $H(t)$; $H'(t) = f(t)$

$$G(x) = [H(t)]_0^{x^2} = H(x^2) - H(0)$$

$$G'(x) = 2x H'(x) - 0$$

(car $u(x) = 0$; $v(x) = x^2$ sont derivable) d'où :

$$G'(x) = 2x f(x^2) = 2x\sqrt{1+x^8}$$

$$\Rightarrow \boxed{G'(x) = 2x\sqrt{1+x^8}}$$

b) Montrons que $\sqrt{1+t^4} \geqslant t^2$ / $t\in\mathbb{R}$:

on a $1+t^4 \geqslant t^4$; $t \longrightarrow \sqrt{t}$ est croissante

donc $\boxed{\sqrt{1+t^4} \geqslant \sqrt{t^4} = t^2}$

6 **Analyse 2** **serie 1**

c) déduisant la limite : $\lim\limits_{x\to+\infty} \frac{G(x)}{x^k}$ pour $0\leq k\leq 4$

on a $G(x)=\int_0^{x^2}\sqrt{1+t^4}\,dt \geqslant \int_0^{x^2} t^2\,dt = \left[\frac{t^3}{3}\right]_0^{x^2} = \frac{1}{3}x^6$

on a $\frac{G(x)}{x^k} \geqslant \frac{1}{3}\frac{x^6}{x^k} = \frac{1}{3}x^{6-k}$; $0\leq k\leq 4$

d'où $\lim\limits_{x\to+\infty}\frac{G(x)}{x^k} \geqslant \lim\limits_{x\to+\infty}\frac{1}{3}x^{6-k} = +\infty$; $6-k\geqslant 0$; $0\leq k\leq 4$

on a $-4\leq -k\leq 0 \Rightarrow 2=6-4\leq 6-k\leq 6$

d'où $\boxed{\lim\limits_{x\to+\infty}\frac{G(x)}{x^k} = +\infty}$

Exercice 9: $F(x)=\int_{1+x}^{x^2+x^3}\frac{dt}{\sqrt{1+t^4}}$ définie sur $\mathbb{R}$

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a) $\boxed{F(1)=\int_2^2\frac{dt}{\sqrt{1+t^4}} = 0}$

b) La dérivabilité de F(x):

on a $t\longrightarrow \frac{1}{\sqrt{1+t^4}}$ continue $= f(t)$

$u(x)=1+x \Rightarrow$ dérivable et $v(x)=x^2+x^3$ dérivable

u' continue

v' continue

$F'(x)=(2x+3x^2)f(x^2+x^3) - 1f(1+x)$

$$\boxed{F'(x)=\frac{2x+3x^2}{\sqrt{1+(x^2+x^3)^4}} - \frac{1}{\sqrt{1+(1+x)^4}}}$$

c) $\lim\limits_{x\to1}\frac{1}{x+1}\int_{1+x}^{x^2+x^3}\frac{dt}{\sqrt{1+t^4}}$

$\Rightarrow \lim\limits_{x\to1}\frac{F(x)}{x+1} = \lim\limits_{x\to1}\frac{F(x)-F(1)}{x-1}$

Par définition; $\boxed{F'(1)=\frac{5}{\sqrt{1+2^4}} - \frac{1}{\sqrt{1+2^4}} = \frac{4}{\sqrt{1+2^4}}}$

Exercice facultatifs:

$\boxed{\begin{array}{l} f : [a;b] \to \mathbb{R} \text{ de classe } \mathcal{C}^1 \text{ c'est à dire:} \\ f : [a;b] \to \mathbb{R} \text{ continue dérivable et } f' \text{ est continue } [a;b] \end{array}}$

- $I_m = \int_a^b f(t) \sin(mt)\, dt$ • $J_m = \int_a^b f(t) \cos(mt)\, dt$

Par partie: $u(t) = f(t)$; $u'(t) = f'(t)$

$v'(t) = \sin(mt)$; $v(t) = \frac{-\cos(mt)}{m}$

$$I_m = \left[-f(t)\frac{\cos(mt)}{m}\right]_a^b + \int_a^b f'(t)\frac{\cos(mt)}{m}\, dt$$

$$I_m = \frac{1}{m}\left[f(a)\cos(ma) - f(b)\cos(mb) + \int_a^b f'(t)\sin(mt)\, dt\right]$$

$$\boxed{\begin{array}{l} * \lim\limits_{m \to +\infty} I_m = 0 \iff \lim\limits_{m \to +\infty} |I_m| = 0 \\ * \lim\limits_{x \to x_0} f(x) = 0 \iff \lim\limits_{x \to x_0} |f(x)| = 0 \end{array}}$$

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$$|I_m| = \frac{1}{m}\left[f(a)\cos(am) - f(b)\cos(bm) + \int_a^b f'(t)\cos(mt)\, dt\right]$$

$$|A + B| \leqslant |A| + |B|$$

$$|I_m| \leqslant \frac{1}{m}\left[|f(a)|\,|\cos(am) + f(b)|\cos(bm)| + \int_a^b |f'(t)|\,|\cos(mt)|\, dt\right.$$

$$\leqslant \frac{1}{m}\left[|f(a)| + |f(b)| + \int_a^b f'(t)\, dt\right]$$

f' est continue sur $[a;b]$ donc $\int_a^b f'(t)\, dt < +\infty$

f continue $\Rightarrow |f(a)| < +\infty$; $|f(b)| < +\infty$

$\Rightarrow |I_m| \leqslant \frac{A}{m}$

$$0 \leqslant \lim_{m \to +\infty} |I_m| \leqslant \lim_{m \to +\infty} \frac{A}{m} = 0 \Rightarrow \lim_{m \to +\infty} |I_m| = 0 \iff \boxed{\lim_{m \to +\infty} I_m = 0}$$